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Describing Function Analysis in the Presence of Uncertainty

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A framework is proposed that generalizes describing function (DF) analysis to uncertain systems. By fitting a rational approximation to the DF of nonlinear elements, DF analysis is incorporated into a generalized μ framework of robustness analysis. This allows us to consider uncertainty in both the linear and nonlinear components. Information on the size, frequency, and stability of unforced limit cycles is obtained as in the graphical test. The response to sinusoidal driving inputs can also be analyzed in this framework.

I. Introduction

COMPUTATIONALLY efficient methods of analysis for nonlinear systems are scarce. In general we are restricted to extensive simulation of the nonlinear model and to linear analysis of different linearizations. Although approximate in nature, linear analysis generally gives good insight into the behavior of the original nonlinear system.

One such approach is harmonic linearization, or sinusoidal input describing function (DF) analysis. This approach is useful in studying the behavior of feedback systems with nonlinear components. In these systems it can predict limit cycle oscillatory behavior reasonably accurately. See, for example, Refs. 1 and 2 for recent applications of these methods. DF analysis is based on finding the roots of a nonlinear function of amplitude and frequency. In many cases, e.g., systems with only one nonlinear component, this can be done graphically in an elegant way; however, graphical describing function analysis technique does not generalize elegantly to systems with uncertainty.

In this paper we will show that by fitting a rational approximation to the DF, limit cycles can be detected in an uncertain system using a recent generalization of the structured singular value μ. Use of the structured singular value paradigm allows for the inclusion of a rich class of uncertainty in the linear component of the system.

As in linear systems robustness analysis, upper and lower bounds are computed for the generalized structured singular value. From these bounds we derive information on the frequency and size of the limit cycles. Using the same quasistationary approximation as in the graphical test, further standard μ tests can be used to determine the stability of the limit cycle. The tool presented here thus completely generalizes the graphical DF analysis test to the case when the linear component has structured uncertainty.

Because the generalized structured singular value plays a central role in other important control problems,[3–5] considerable effort is being devoted to the development of efficient numerical algorithms to compute it. Recasting the DF analysis problem into this setup will allow us to benefit from that effort.

The paper is organized as follows. In the next section, we will briefly review the generalization of the structured singular value μ theory and show how to represent a rational nonlinear function of a signal u as a constrained uncertain linear system. In Sec. III, after reviewing the standard DF analysis technique, we will show how to extend it to the case where the linear component is uncertain. In Sec. IV, we will develop and analyze an example to show the validity of the tool. Finally, we will present conclusions in Sec. V.

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II. Preliminaries

The reader is assumed to be familiar with the structured singular value or μ analysis setup. For a good tutorial on μ, see Ref. 6.

A. Generalized Structured Singular Value genμ

The structured singular value of a matrix A and a block structure $\boldsymbol{\Delta}$ measures the minimum size of an element of $\boldsymbol{\Delta}$ that makes the matrix $I - M\Delta$ singular. If we denote this value by μ, the system of equations

$$v = Mx \qquad x = \Delta v \tag{1}$$

admits only the zero solution for all $\Delta \in \boldsymbol{\Delta}$ with $\|\Delta\| < \mu$.

We can define a more general value by restricting the directions in which we care about $I - M\Delta$ being singular. Consider the system of equations

$$v = Mx \qquad x = \Delta v \qquad 0 = Cx \tag{2}$$

To determine whether this system admits more than one solution, we make the following definition.

Definition 1 (Refs. 4 and 5): The generalized structured singular value (genμ) of the pair (A, B) and block structure $\boldsymbol{\Delta}$ is defined as follows.

If, for all $\Delta \in \boldsymbol{\Delta}$,

$$\ker \begin{bmatrix} I - \Delta M \\ C \end{bmatrix} = \{0\}$$

then

$$\text{gen}\mu_{\boldsymbol{\Delta},C} := 0$$

otherwise

$$\text{gen}\mu_{\boldsymbol{\Delta},C} := \left(\min \left\{ \bar{\sigma}(\Delta) : \Delta \in \boldsymbol{\Delta},\ \ker \begin{bmatrix} I - \Delta M \\ C \end{bmatrix} \neq \{0\} \right\} \right)^{-1}$$

Then it will hold that the system of equations (2) admits only the zero solution for all $\Delta \in \boldsymbol{\Delta}$ with $\|\Delta\| < \text{gen}\mu$.

A lower bound on genμ can be computed using standard constrained optimization methods. To compute an upper bound, we will use the following theorem.

Theorem 1 (Ref. 4), $\text{gen}\mu(M, C) \leq \min_{Q \in C^{n\times m}} \mu(M + QC)$*:* In addition to the fact that we can solve exactly for the Q that minimizes the standard μ upper bound of $\mu(M + QC)$, Theorem 1 allows us to compute an upper bound on genμ as the solution of an affine matrix inequality. See Refs. 4 and 7 for more details.

B. Rational Functions as Linear Fractional Transformations with Constraints

Any rational expression of δ bounded at the origin can be expressed as a linear fractional transformation (LFT) between a constant matrix M containing the coefficients of the rational function and a diagonal matrix δI whose size depends on the order of the expression. In the standard μ setup, these rational functions are used as multiplicative transfer functions. In this case, δ represents either an uncertain parameter or the frequency variable. When the input is u, the output of the system is given as the solution of the system of equations

$$\begin{bmatrix} \zeta \\ y \end{bmatrix} = M \begin{bmatrix} \xi \\ u \end{bmatrix} \qquad \xi = \delta I_n \zeta \tag{3}$$

or using the Redheaffer star product notation,

$$y = (M \star \delta I)u$$

Now consider the rational function of u, $y = R(u)$. The following lemma shows how this function can also be represented as a multiplicative LFT by introducing an additional parameter δ and a constraint.

Lemma 1: Consider a rational function $y = R(u)$ and its corresponding LFT representation $M \star \delta I_n = R(\delta)$. Then y is the solution to the following system of equations:

$$\begin{bmatrix} \zeta_1 \\ \zeta \\ y \end{bmatrix} = N \begin{bmatrix} \xi_1 \\ \xi \\ u \\ 1 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} \xi_1 \\ \xi \end{bmatrix} = \delta I_{n+1} \begin{bmatrix} \zeta_1 \\ \zeta \end{bmatrix} \tag{5}$$

$$0 = C \begin{bmatrix} \xi_1 \\ \xi \\ u \\ 1 \end{bmatrix} \tag{6}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0_{1\times n} & 0 & 1 \\ 0_{n\times 1} & M_{11} & M_{12} & 0 \\ 0 & M_{21} & M_{22} & 0 \end{bmatrix}$$

$$C = [1 \quad 0_{1\times n} \quad 0 \quad -1]$$

and $M_{11} \in R^{n\times n}$.

Proof: From the definition of N, it follows that $\zeta_1 = 1$, and thus from Eq. (5),

$$\xi_1 = \delta$$

Eq. (6) can then be rewritten as

$$\delta - u = 0 \tag{7}$$

Finally, from the definition of M we have

$$y = R(\delta) \tag{8}$$

Lemma 1 follows from Eqs. (7) and (8). □

Remarks: Equations (4–6) can be represented graphically as in Fig. 1. The multiplicative nature of this representation is evident in this diagram.

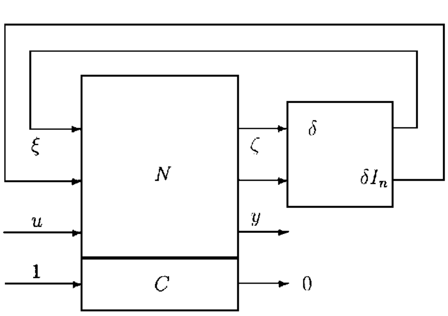

Fig. 1 Rational function as a multiplicative LFT.

Because the composition of LFTs is another LFT, we can substitute δ in all of the preceding equations by any rational expression of another variable δ'. A particular application of this transformation is to replace Eq. (7) with an equation of the form

$$\frac{u_{\max} + u_{\min}}{2} + \frac{u_{\max} - u_{\min}}{2}\delta - u = 0$$

In this case, $|\delta| \le 1$ if and only if $u_{\min} \le u \le u_{\max}$.

III. Describing Function Analysis as a Generalized μ Test

DF analysis allows us to study the behavior of the system in the frequency domain when certain assumptions hold. Because the system is nonlinear, this analysis is necessarily an approximation; however, in practice it has proven useful in predicting limit cycle behavior of nonlinear feedback systems. In this section, we will first briefly review the DF analysis methods and the assumptions necessary for its validity. Next, we will show how to recast this analysis method into the μ analysis framework. This will allow us to include uncertainty in the linear component in a systematic way.

A. Sinusoidal Input Describing Function Analysis

Consider the feedback interconnection of Fig. 2. We would like to know if this interconnection admits periodic solutions. First, we make a definition and a series of assumptions.

Definition 2—DF: The DF is the complex, fundamental-harmonic gain of a nonlinearity in the presence of a driving sinusoid.

Assumptions 1—Sinusoidal DF Analysis (Ref. 8): Consider the feedback interconnection of Fig. 2. We will assume that the following hold.

1) If the input to the nonlinear element is sinusoidal, the output of the nonlinear element is periodic and of the same fundamental frequency as the input.

2) Only the fundamental of the output wave need be considered in a frequency response analysis.

3) The nonlinear element is time invariant.

4) Only one nonlinear element is considered to exist in the system.

If these assumptions hold, we can replace the nonlinear component with its DF and perform the analysis in the frequency domain. Because the DF is amplitude dependent, this analysis includes an amplitude as well as a frequency search. Several graphical methods can be used to perform this search[8]; however, these methods are not easily generalized to the case where the DF is frequency dependent or where there is uncertainty in the linear components.

The answers provided by DF analysis are limited. It simply tries to establish whether solutions of a particular form exist. It is entirely possible for this analysis to predict no limit cycles, but such limit cycles exist, although they are not sinusoidal. Also, several solutions might be found, not all of which are stable, and the small signal model around those solutions may or may not be stable. This by no means makes the approach useless; however, the burden on the designer to interpret and to evaluate the answers obtained remains.

Despite these limitations, DF analysis has proven useful, especially because it gives good insight into the effect the variation of different parameters may have on the behavior of the system. For a more complete discussion on the uses and limitations of DF analysis, see Ref. 9.

B. DF Analysis in the Presence of Uncertainty

Using the elements described in Sec. II, we will now build a methodology for doing DF analysis when there is uncertainty in the linear component. To systematize the analysis of this kind of system, we will first state a more specific performance requirement.

Question 1—Worst Case Limit Cycles: Given an uncertain linear system of the form

$$G(j\omega) = P(j\omega) \star \Delta$$

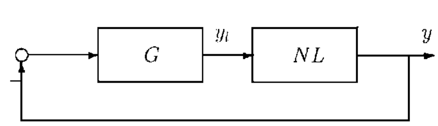

Fig. 2 Interconnection for performance analysis.

where Δ is a complex block diagonal matrix describing the uncertainty as done in the standard μ analysis setup,[6] connected in a feedback loop with a nonlinear component as shown in Fig. 2, we would like to answer the following question: If Assumptions 1 hold, and if the signal y_l remains bounded in the sense $|y_l| < 1$, will y remain bounded in the sense $|y_o| < 1$ where y_o is the first harmonic component of y for all Δ with the given structure verifying $\bar{\sigma}(\Delta) \le 1$, and for all ω?

Remarks: The bounds on the input and outputs do not have to be the same. For simplicity, we will choose them to be equal to one for the development that follows but they can be chosen arbitrarily.

The performance Question 1 asks whether or not limit cycles larger than a given size will be established. Requiring that y_l remain bounded is a technical necessity for the development of the methodology. A coarse bound on the size of y_l can be obtained from classical analysis of the nominal system and, in general, from an understanding of the physical nature of the signals involved. Also, for all common nonlinearities, outside a certain range of the input the DF is a constant (making the overall system linear in that range). We can use this range to bound y_l for the DF analysis and study the behavior of the system with constant feedback outside this range (see the example in Sec. IV).

The system considered does not have to be single input/single output; the nonlinear component, however, has to have one single input.

We will now show how to answer Question 1 by computing a generalized structured singular value. We can only do this in cases where the DF of the nonlinearity can be approximated by a rational function of the amplitude and frequency of the input signal.

First, note that the signal y_l can be assumed to be real and positive without loss of generality by appropriately choosing the origin of time. If Assumptions 1 hold, we can replace the nonlinear system by its sinusoidal input DF and the describing function by its rational approximation.

We can now proceed to replace the rational function block by its corresponding LFT representation, as described in Sec. II.B. The feedback loop is now as in Fig. 3.

A feedback loop made of LFT blocks can be rewritten as one single LFT between a larger matrix and an uncertainty structure built by combining the individual uncertainty components in block diagonal form. The transfer function in Fig. 3 can then also be represented as in Fig. 4.

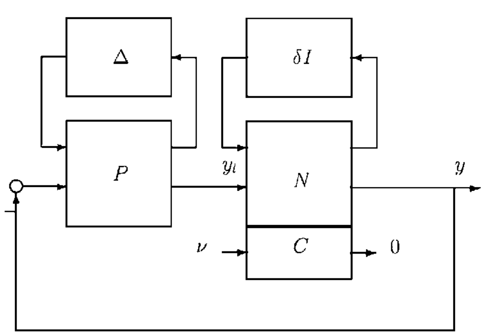

Fig. 3 Feedback loop as interconnection of LFTs.

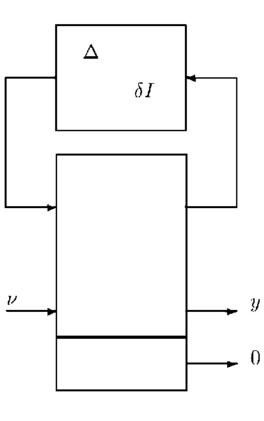

Fig. 4 LFT interconnection.

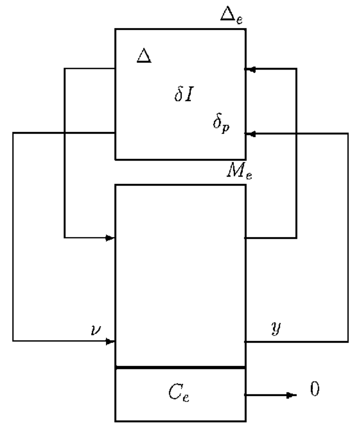

Fig. 5 Performance analysis interconnection.

Finally, we must add to the uncertainty structure the blocks needed to impose the desired performance condition. For this purpose, we augment the uncertainty structure with one additional block δ_p connected as shown in Fig. 5. This figure represents the set of equations

$$y_l = (P \star \Delta)(-y) \qquad y = (N \star \delta I)\begin{bmatrix} y_l \\ \nu \end{bmatrix}$$
$$\nu = \delta_p y \qquad 0 = y_l - 0.5\delta\nu - 0.5\nu \tag{9}$$

together with the constraints

$$\bar{\sigma}(\Delta) \le 1 \Delta \in \boldsymbol{\Delta} \qquad |\delta| \le 1 \delta \in \boldsymbol{R} \qquad |\delta_p| \le 1 \delta_p \in \boldsymbol{C} \tag{10}$$

We will first prove a technical lemma needed for the main result in this paper.

Lemma 2: Assume the uncertain linear system G is well posed for all frequencies and for all Δ and δ verifying the constraints (10). Then if there is a nonzero solution to the system of equations (9) and constraints (10), there is a solution with $\nu = 1$.

Proof: The constraint

$$0 = y_l - 0.5\delta\nu - 0.5\nu$$

implies that $y_l = 0$. As the LFT describing the rational function is assumed to be well posed for all values of δ, we will then have $y = 0$. Finally, the input to the linear component is $u - y = 0$, and because this component is assumed to be well posed for all Δ and δ, then all signals in the diagram are zero. A nontrivial solution then has $\nu \ne 0$. Fix the values of Δ, δ_p, and δ. Then equations (9) and constraints (10) are linear in ν. If there is a solution with $\nu \ne 0$, there is a solution with $\nu = 1$. □

We are now ready to state the main result of this paper.

Theorem 2: Assume the uncertain linear system G is well posed for all frequencies and for all Δ and δ verifying the constraints (10). Then the following statement—1) the answer to Question 1 is yes; 2) the system of equations (9) and constraints (10) represented by Fig. 5 admits only the zero solution; and 3) gen$\mu_{\Delta_e}(M_e, C_e) < 1$—are equivalent.

Proof: The equivalence of statements 2 and 3 follows from Definition 1.

$1 \Rightarrow 2$ We will prove the counterpositive not2 $\Rightarrow$ not1. Assume there is a nonzero solution to equations (9) and constraints (10). Then according to Lemma 2 there is a solution with $\nu = 1$. In this case, Figs. 3 and 4 represent the same equations. Furthermore $|\delta_p||y| = 1$, which implies $|y| \le 1$, and thus the answer to Question 1 is no.

$2 \Rightarrow 1$ We will prove the counterpositive not1 $\Rightarrow$ not2. Assume the answer to Question 1 is no. Then there exists y_l, y with $|y| > 1$ and $y_l \le 1$ and real that verify the equations in diagram 2. Setting $\delta_p = 1/y$ and $\nu = 1$, then equations (9) will admit a nonzero solution verifying constraints (10). □

Remarks: More complex interconnections than the one in Fig. 2 can be used. The signal ν can have dimensionality larger than 1, and a dynamical component can be placed following the nonlinearity or in the feedback loop. The only restriction is that all linear systems are well posed at all frequencies.

As with standard DF analysis, this test will only detect the establishment of sinusoidal oscillations. Further analysis is required to determine if those oscillations are stable or if some other form of instability is possible.

IV. Example

To evaluate the behavior of the aforementioned bounds, we consider the following example. The linear component of the loop is given by the transfer function

$$G(s) = \frac{0.6a\omega_0^2}{(s+a)\left(s^2 + 2\delta\omega_0 s + \omega_0^2\right)}$$

where

$$a = 10 \qquad \omega_0 = 2 \qquad \delta = 0.1$$

We will assume a multiplicative model for the uncertainty. We will study two levels of uncertainty, 1 and 10% (corresponding to $W_d = 0.1$ and 0.01 in Fig. 6). The nonlinear element will be a backlash of deadband width one and slope one. The linear and nonlinear components are interconnected in the feedback loop shown in Fig. 6. We will denote by N the DF of this nonlinearity. As a function of the quotient x of the input amplitude and the deadband width, the DF corresponding to a backlash nonlinearity is

$$N(x) = (1/\pi)\big((\pi/2) + \sin^{-1}[(x-1)/x] + [(x-1)/x]$$
$$\times \cos\{\sin^{-1}[(x-1)/x]\} + (i/\pi x)[(1/x) - 2]$$

for x larger than 0.5 and 0 otherwise. We will use the following rational approximation in the nonzero interval:

$$Nr(x) = (1/\pi)[(\pi/2) - 2y + (y^3/3)] + (i/\pi)(y^2 - 1)$$

where

$$y = (1/x) - 1$$

We can verify that

$$\lim_{x\to\infty} |N(x)| = 1$$

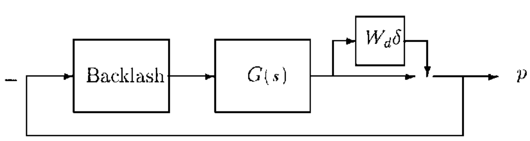

Fig. 6 Loop interconnection.

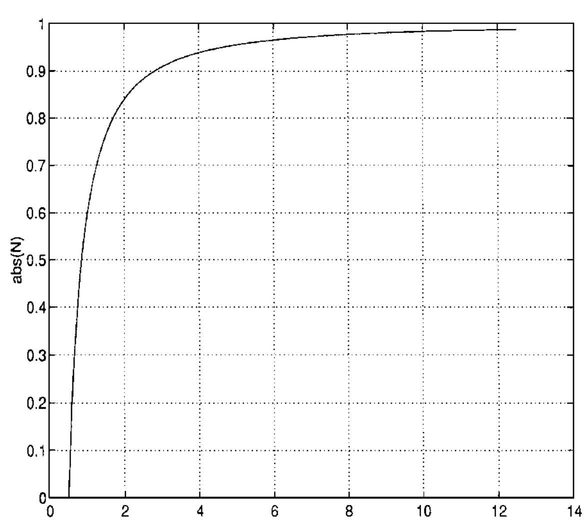

Fig. 7 Magnitude of backlash DF.

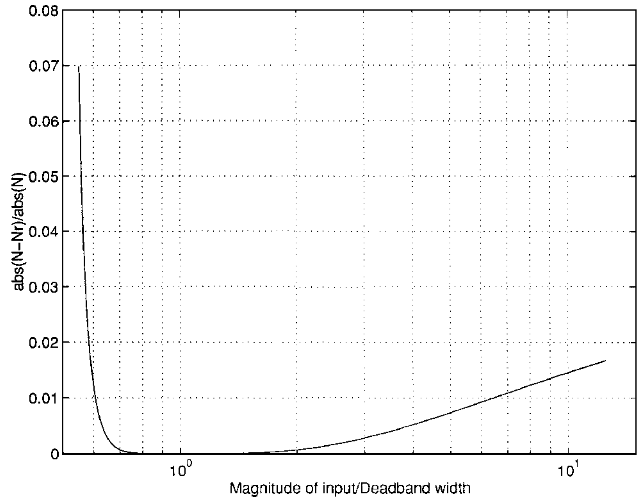

Fig. 8 Approximation error for backlash DF.

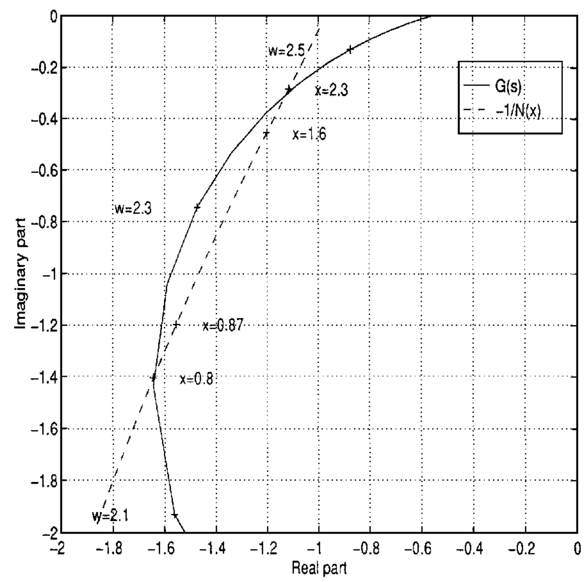

Fig. 9 Nyquist diagram, detail of intersections.

(see Fig. 7) and that

$$|N(x)| > 0.98 \qquad \text{for} \qquad x \geq 12.5$$

$$|N(x)| < 0.12 \qquad \text{for} \qquad 0.57 \geq x$$

We will work in the interval $0.57 \leq x \leq 12.5$. Figure 8 shows the approximation error in the DF for this range. For $x > 12.5$, the DF is flat and thus very susceptible to error in determination of the limit cycle size. The existence of instabilities can be detected by studying the robust stability of the closed-loop system with constant feedback 1. For values of x in the range $0.5 \leq x \leq 0.57$, robust stability can also be determined closing the loop with a norm bounded perturbation in place of the nonlinear element. If this system is unstable, further analysis could be done with a different rational approximation of the DF accurate around $x = 0.5$.

To compare the proposed approach with the current DF analysis, in Fig. 9 we show the Nyquist plot of $G(s)$ superimposed with $-1/N$. There are two intersections of the two curves. The top one corresponds to a stable limit cycle, the bottom one to an unstable limit cycle. The performance measure will be the magnitude of the signal p. We compute upper and lower bounds on the magnitude of p for a set of frequency intervals.

The lower bound is computed as an optimization problem, namely, finding the frequency and perturbation in the given intervals that maximize the performance norm. The frequency domain upper bound is computed at the frequency determined by the lower

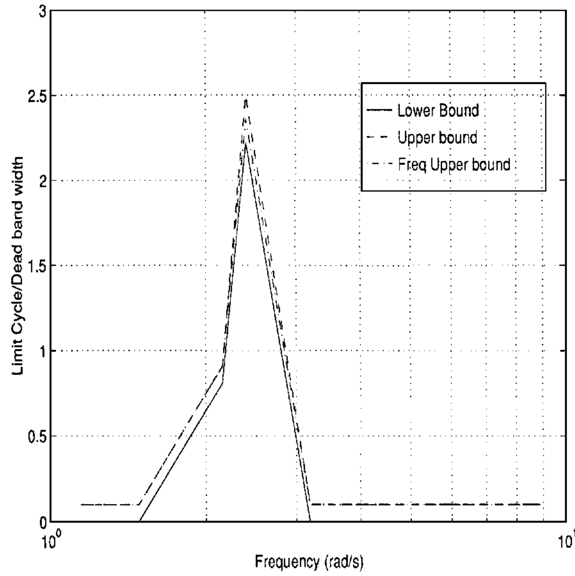

Fig. 10 Performance measure, 1% uncertainty.

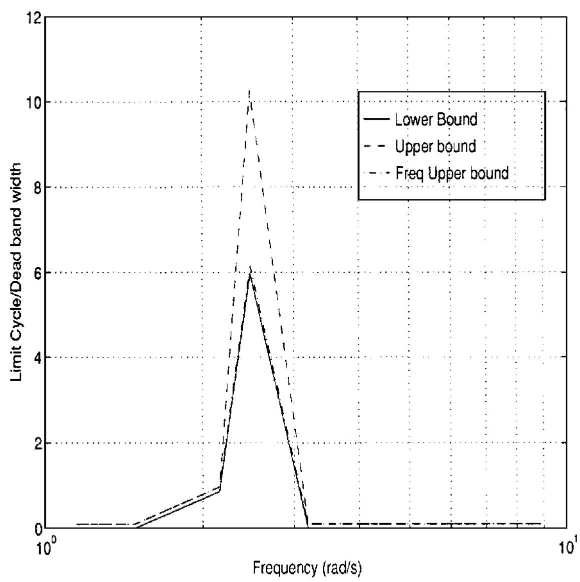

Fig. 11 Performance measure, 10% uncertainty.

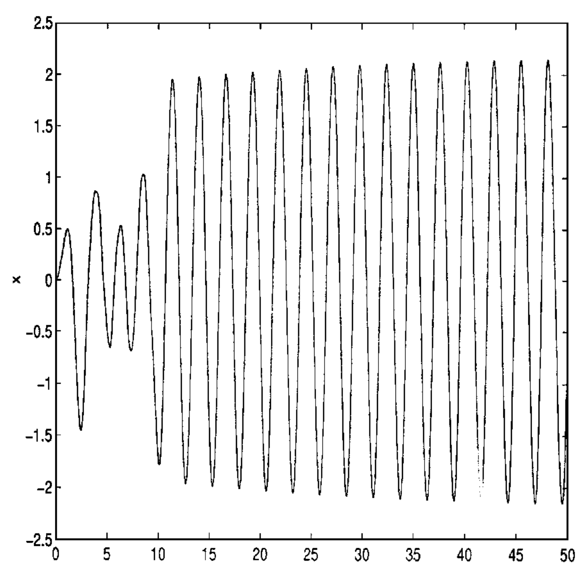

Fig. 12 Nonlinear simulation, no uncertainty. The observed limit cycle frequency is 2.4 rad/s.

Table 1 Comparison of predicted and simulated limit cycles

	Nominal	Predicted by DF	Predicted by DF with uncertainty	Perturbed model
Frequency, rad/s	2.4	2.4	2.4	2.4
Amplitude	2.1	2.3	$6.0 \leq x \leq 6.1$	7.2

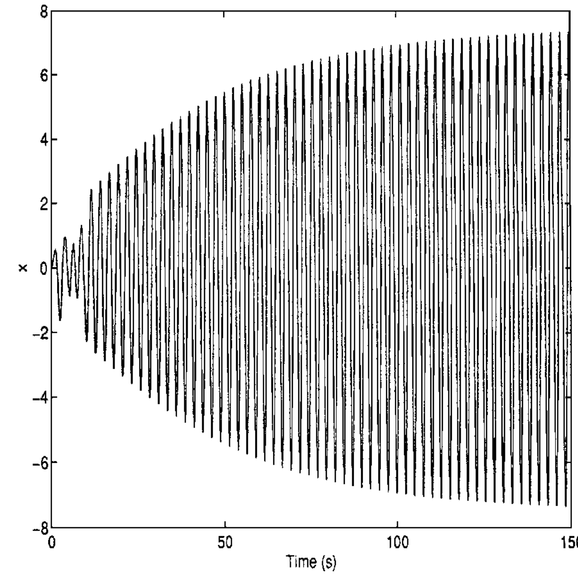

Fig. 13 Nonlinear simulation, 10% perturbation. The observed limit cycle frequency is 2.38 rad/s.

bound algorithm. The state-space upper bound is computed over the whole interval. The results obtained for the two uncertainty levels are summarized in Figs. 10 and 11. For simplicity, all bounds are plotted at the frequency where the lower bound is achieved.

Finally, to validate the results, we ran simulations of the system considered with no uncertainty and with the perturbation selected by the lower bound algorithm at the 10% level. The results obtained are shown in Figs. 12 and 13. During the first 100 s, the system is externally perturbed to start the limit cycles. The worst-case perturbation was complex, and so it was implemented with a low-pass filter having the desired value at the frequency of the predicted oscillations. The time domain simulations match the predicted oscillations accurately, as summarized in Table 1.

V. Conclusions

In cases where a good rational approximation of the DF can be obtained, and when a bounded interval of interest can be identified in the input to the nonlinear element, we showed that performance analysis on the harmonic linearization of the system can be done with a generalized μ test. In this case, the linear component of the feedback system can be uncertain. We can use the rich description of uncertainty allowed by the μ setup: unmodeled dynamics and real and complex uncertain parameters. It is also possible to consider multiple-input/multiple-output systems in this setup; however, the nonlinearity is restricted to be unique and single input/single output. The test presented provides similar information on the size, frequency, and stability of the limit cycles as does the classic graphic DF analysis test.

The applicability of the test was demonstrated through a numerical example. This example, together with others carried out but not reported here due to space limitations, show good correlation between the predicted and observed limit cycles. They also show that moderate amounts of uncertainty can produce significant changes in limit cycle size. This fact justifies the added complexity of using this tool as compared with traditional graphical DF analysis.

Computation time for the most complex example treated was on the order of 8 h running under MATLAB, on an SGI Indigo 2, for 10 frequency points. There is, however, significant room for improvement, as we used off-the-shelf software without any customization. As the computation algorithms for the upper and lower bounds to the generalized structured singular value achieve the maturity of the corresponding ones for the standard μ problems, the computation burden of this test will be reduced. As the same test is used for other applications in robust control, a significant research program is being carried out in this area.

Acknowledgments

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